WEAK DOMINATION IN SEMITOTAL BLOCK GRAPHS

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ABSTRACT: For any graph $G=(V,E)$, the semitotal block graph $T_b(G)=H$ whose set of vertices is the union of the set of vertices and blocks of G and in which two vertices are adjacent if and only if the corresponding vertices of G are adjacent or the corresponding members are incident in G. A dominating set S of G is said to be weak dominating set, if every vertex $u \epsilon V-S$ is adjacent to a vertex $v \epsilon S$, such that $\deg(v) \leq \deg(u)$. A dominating set D of a graph H is a weak dominating set of H, if every vertex in $V[H]-D$ is weakly dominated by at least one vertex in D. A weak semitotal block domination number $\gamma_{wtb}(G)$ of G is the minimum cardinality of a weak semitotal block dominating set of G. In this paper, we obtained some shorp bounds for $\gamma_{wtb}(G)$. Also some upper and lower bounds on $\gamma_{wtb}(G)$, in terms of elements of G and other dominating parameters of G were obtained.

Keywords: Graph, Line graph, Domination number, Weak domination number, Weak semitotal block domination number.

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1. INTRODUCTION

All the graphs consider here are finite, without loops and multiple edges, undirected and connected. Graph terminology not found here can be found in [6]. Specifically, let $G=(V,E)$ be a graph with vertex set V and edge set E, such that $|V|=p$ and $|E|=q$. Degree of an edge $=uv$ of a graph G is defined as $\deg(u)+\deg(v)-2$ and is denoted as $\delta'(G)$. In general we use $<X>$ to denote the subgraph induced by the set of vertex X and $N(v)$ and $N[v]$ denote the open and closed neighborhood of a vertex v, respectively. The minimum (maximum) degree among the vertices of G is denoted by $\delta(G)[\Delta(G)]$. Also $\beta_0(G)$ $[\beta_1(G)]$ is the minimum number of vertices (edges) in a maximal independent set of vertex(edge) of G. A vertex cover in a graph G is a set of vertices that covers all the edges of G. The vertex covering number $\alpha_0(G)$ is the minimum cardinality of a vertex cover in G. An edge cover of a graph G without isolated vertices is a set of edges of G that covers all the vertices of G. An edge covering number $\alpha_1(G)$ is the minimum cardinality of an edge cover in G.

The minimum number of colors in any coloring of a graph G such that no two adjacent vertices have the same color is called the chromatic number of G and is denoted by $\chi(G)$.

A set $D \subseteq V(G)$ is a dominating set, if for every vertex $v \epsilon V(G)-D$ is adjacent to at least one vertex $u \in D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A thorough study of domination appears in [8, 9].

We begin by recalling some standard definitions from domination theory.

A set $S \subseteq V(G^2)$ is a dominating set, if each vertex in $V(G^2)-S$ has one neighbor in S. The square domination number of G is denoted by $\gamma(G^2)$ is the minimum cardinality of a square dominating set of G^2. This concept was introduced by [18].

A dominating set $D \subseteq V(G)$ is a connected dominating set, if the induced subgraph $< D >$ has one component. The connected domination number $\gamma_c(G)$ is the minimum cardinality of a connected dominating set of G.

Further, a set $F \subseteq E(G)$ is called an edge dominating set, if for every edge $e \epsilon E(G) - F$ is adjacent to at least one edge $f \in F$. An edge domination number $\gamma'(G)$ is the minimum cardinality of an edge dominating set of G, see [1].

En edge dominating set $F \subseteq E(G)$ is said to be connected edge dominating set, if the induced subgraph $< F >$ is connected. An edge connected domination number γ_c'(G) is the minimum cardinality of an edge connected dominating set of G, for details see [2].

A dominating set $S \subseteq V(G)$ is called an end dominating set, if S contains all the endvertices of G. An end domination number $\gamma_e(G)$ is the minimum cardinality of an end dominating set of G. Domination related parameters were now studied in graph theory [8, 9, 16].

The concept of perfect domination was introduced and studied in [4]. A dominating set $D \subseteq V(G)$ is said to be perfect dominating set, if for every element $v \epsilon V(G) - D$ is dominated by exactly one element $u \in D$. The perfect domination number $\gamma_p(G)$ is the minimum cardinality of a perfect dominating set of G.

A dominating set $S \subseteq V(G)$ is called a double dominating set, if every vertex of $V(G)$ is dominated by at least two vertices in S. The double domination number $\gamma_{dd}(G)$ is the minimum cardinality of a double dominating set of G. The concept of double domination was introduced in [7].

In [3], the author showed that a Roman domination function of a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of Roman domination function in G is the value of $f(v) = \sum_{u \in v} f(u)$. The Roman domination number of G is denoted by $\gamma_R(G)$, equals the smallest weight of a Roman dominating function on G.

The concept of restrained domination in graphs was introduced by Domke et. at (1999) see [5]. A set $S \subseteq V(G)$ is a restrained dominating set of G, if every vertex $V(G) - S$ is adjacent to a vertex in S and another vertex in $V(G) - S$. The restrained domination number of a graph G is denoted by $\gamma_{Res}(G)$ is the minimum cardinality of a restrained dominating set of G.

Analogously, a set D_ε of elements of G is an entire dominating set, if every element not in D_ε is either adjacent or incident to at least one element in D_ε. The entire domination number of G is denoted by $\gamma_\varepsilon(G)$ of G is the minimum cardinality of an entire dominating set of G, see[11].

The concept of split domination number introduced by [10]. A dominating set $D \subseteq V(G)$ is a split dominating set, if the induced subgraph $< V - D >$ has more than one component. The split domination number of G is denoted by $\gamma_s(G)$ is the minimum cardinality of a split dominating set of G.

A dominating set $S \subseteq V(G)$ is called a nonsplit dominating set, if the induced subgraph $< V - S >$ is connected. The nonsplit domination number $\gamma_{ns}(G)$ is the minimum cardinality of a nonsplit dominating set of G. For details see [10].

A dominating set $D \subseteq V(G)$ is a strong nonsplit dominating set, if the induced subgraph $< V - D >$ is complete. The strong nonsplit domination number of a graph G is denoted by $\gamma_{sns}(G)$ is the minimum cardinality of a strong nonsplit dominating set of G, see [10].

In [19], Sampathkumar and L. Pushpa Latha have shown weak domination number. A dominating set S is a weak dominating set of G, if for every vertex $u \in V(G) - S$ there is a vertex $v \in S$ with

$\deg(v) \leq \deg(u)$ and u is adjacent to v. A weak domination number $\gamma_w(G)$ is the minimum cardinality of a weak dominating set of G.

A weak dominating set D is a weak dominating set of $L(G)$, if for every vertex $u \in V[L(G)] - D$ there is a vertex $v \in D$ with $\deg(v) \leq \deg(u)$ and u is adjacent to v. A weak line domination number $\gamma_{wl}(G)$ is the minimal cardinality of a weak line dominating set of G, see [12].

A weak dominating set S is a weak dominating set of $B(G)$, if for every vertex $u \in V[B(G)] - S$ there is a vertex $v \in S$ with $deg(v) \leq \deg(u)$ and u is adjacent to v. The weak block domination number $\gamma_{wb}(G)$ of G is the minimum cardinality of a weak dominating set of $B(G)$. This concept is discussed in [13].

The purpose of this paper is to introduced the concept of weak domination in semitotal block graphs and study its properties.

A dominating set D of a graph $T_b(G)$ is a weak semitotal block dominating set of G. If every vertex $u \in V[T_b(G)] - D$ is weakly dominated by at least one vertex $v \in D$ with $\deg(v) \leq \deg(u)$ and u is adjacent to v. The weak semitotal block domination number of G is denoted by $\gamma_{wtb}(G)$ is the minimum cardinality of a weak semitotal block dominating set of G.

Further domination related graph valued functions has been studied in [14, 15, 17].

2. MAIN RESULTS

We develop the following results for some standard graphs.

Theorem 1: a]. For any cycle C_p with $p \geq 3$ vertices, then

$$\gamma_{wtb}[C_p] = \begin{cases} \frac{p}{3} \; if \; p \equiv 0(mod\ 3) \\ \left\lceil \frac{p}{3} \right\rceil \; Otherwise. \end{cases}$$

b]. For any star $K_{1,n}$ $n \geq 1$, then

$$\gamma_{wtb}[K_{1,n}] = \Delta(G).$$

c]. For any complete graph K_p with $p \geq 2$ vertices, then

$$\gamma_{wbt}[K_p] = 1.$$

d]. For any path P_p with $p \geq 2$ vertices, then

$$\gamma_{wtb}[P_p] = diam(G).$$

Theorem 2: For any nontrivial tree T, then $\gamma_{wtb}(T) = q$.

Proof: For any nontrivial tree T, since each edge is a block, then in $T_b(T)$ these are the block vertices. Also these block vertices of $T_b(T)$ are the elements of a weak dominating set of $T_b(T)$. Hence $\gamma_{wtb}(T) = q$.

Theorem 3: For any nontrivial tree T, with l endvertices and s support vertices, then $\gamma_{wtb}(T) = s + l - \delta$ if and only if all the nonendvertices of a tree T is adjacent to at least one endvertex of T.

Proof: For the necessary, suppose T has at least one nonendvertex v which is not adjacent to an endvertex. Then by Theorem 2,$\gamma_{wtb}(T) \neq s + l - \delta$.

For the sufficiency, for any tree T with $p \geq 2$ vertices with s number of support vertices and l number of endvertices the total number of block vertices in $T_b(T)$ is $s + l - \delta$. Since for any tree T with $p \geq 2$ vertices each edge is a block and the weak domination in $T_b(T)$ is $\{s + l - \delta\}$. Hence $\gamma_{wtb}(T) = s + l - \delta$.

Theorem 4: For any connected (p, q) graph G, with $p \geq 2$ vertices, then $\gamma_{wtb}(G) \leq \gamma_{dd}(G) + \Delta(G) + 3$.

Proof: Let $D = \{v_1, \dots, v_n\} \subseteq V(G)$ be the set of nonendvertiecs with $\deg(v_i) \geq 2$, $\forall\, v_i \in D$, $1 \leq i \leq n$, such that $N[D] = V(G)$, then D forms a minimal dominating set of G. Let $V_1 = V(G) - D$ and $D_1 = \{v_1, \dots, v_m\} \subseteq V_1$, then $D \cup D_1$ forms a double dominating set of G. For any graph G, there exists at least one vertex $v \in V(G)$ with $\deg(v) = \Delta(G)$. Let $A = \{v_1, \dots, v_p\}$ be the block vertices in $T_b(G)$. Suppose $B = \{v_1, \dots, v_q\}$ be the set of all endvertices in G and $B_1 = \{v_1, .., v_l\}$ be the set of vertices adjacent to the endvertices of B in $T_b(G)$.

In $T_b(G)$, $\forall\, v_j \in V[T_b(G)] - \{A - B_1\} \cup B$ is adjacent at least one vertex of v_k, $\forall\, v_k \in \{A - B_1\} \cup B$, such that $\deg(v_k) \leq \deg(v_j)$. Hence $S = \{A - B_1\} \cup B$ forms a minimal $\gamma_{wtb} - set$ of G. Thus $|S| \geq |D \cup D_1| + |\deg(v)| + 3$, gives $\gamma_{wtb}(G) \leq \gamma_{dd}(G) + \Delta(G) + 3$.

Theorem 5: For any tree T, with $p \geq 4$ vertices, then $\gamma_{wtb}(T) \geq \gamma_{Res}(T) + \gamma(T) - 1$ and $T \neq K_{1,n}$.

Proof: Suppose $T = K_{1,n}$. Then $\gamma_{wtb}(T) \ngeq \gamma_{Res}(T) + \gamma(T) - 1$. Let $C = \{v_1, \dots, v_n\} \subseteq V(T)$ be the set of all cutvertices in T. Suppose $C_1 \subseteq C$, such that $N[C_1] = V(T)$. Then C_1 forms a minimal dominating set of T. Let $D = \{v_1, \dots, v_m\}$ be the set of all endvertices in T. Further, if $\forall\, v_j \in V(T) - \{D \cup C_1\}$ is adjacent to least one vertex of $v_i \in \{D \cup C_1\}$ and at least one vertex of $V(T) - \{D \cup C_1\}$. Then $S = D \cup C_1$ forms a $\gamma_{Res} - set$ of T. Let $R = \{u_1, \dots, u_p\} \subseteq V[T_b(T)]$ be the set of vertices with $\deg(u_i) = 2$, $\forall\, u_i \in R$. Further, let $R_1 = \{u_1, \dots, u_q\} \subseteq R$, such that $N[R_1] = V[T_b(T)]$. Then R_1 forms a minimal dominating set of $T_b(T)$.

Suppose, $\forall\, u_j \in V[T_b(T)] - R_1$ is adjacent to at least one vertex of $u_i \in R_1$ with $\deg(u_i) \leq \deg(u_j)$ and u_j is adjacent to u_i. Hence R_1 forms a minimal $\gamma_{wtb} - set$ of T. Thus $|R_1| \geq |S| + |C_1| - 1$, gives $\gamma_{wtb}(T) \geq \gamma_{Res}(T) + \gamma(T) - 1$.

In the following theorem we obtained lower bound for $\gamma_{wtb}(G)$ in terms of weak line domination number and minimum edge degree of G.

Theorem 6: For any connected (p, q) graph G, with $p \geq 3$ vertices, then $\gamma_{wtb}(G) + \delta'(G) \geq \gamma_{wl}(G) + 2$.

Proof: Let $H = \{v_1, \dots, v_n\} \subseteq V[L(G)]$ be the set of vertices with $\deg(v_i) \geq 1$, $\forall\, v_i \in H$, $1 \leq i \leq n$, such that $N[H] = V[L(G)]$. Then H forms a minimal dominating set of $L(G)$.Suppose there exists a set $H_1 \subseteq V[L(G)] - H$, such that $\forall\, v_j \in H, \deg(v_i) \leq \deg(v_j), \forall\, v_i \in H_1$ and v_i is adjacent to v_j. Then $\{H \cup H_1\}$ forms a minimal $\gamma_{wl} - set$ of G. Let $e \in E(G)$ with $deg(e) = \delta'(G)$.

Assume M be the set of endvertices.

Now we consider the following cases.

Case 1: Suppose $M = \phi$. Let $\{B_1, \dots, B_K\}$ be the number of blocks in G. In $T_b(G)$, let $A = \{v_1, \dots, v_n\}$ be the set of block vertices corresponding to the blocks of G. Suppose $A_1 = \{v_1, \dots, v_m\} \subset A$. Further, let $B = V[T_b(G)] - A$ be the set of vertices with minimum degree, such that $\forall\, v_j \in V[T_b(G)] - \{A_1 \cup B\}$ is adjacent to at least one vertex of $v_i \in \{A_1 \cup B\}$ with $\deg(v_i) \leq \deg(v_j)$ and v_j is adjacent to v_i. Then $D = \{A_1 \cup B\}$ forms a $\gamma_{wtb} - set$ of G. Hence $|D| + |\deg(e)| \geq |\{H \cup H_1\}| + 2$, gives the result.

Case 2: Suppose $M \neq \phi$, $let\ C = \{v_1, \dots, v_p\}$ be the set of all endverties in G and $C_1 = \{v_1, \dots, v_q\}$ be the set of vertices adjacent to the endvertices of C in $T_b(G)$. Then $\forall\, v_l \in V[T_b(G)] - \{D \cup C_1\}$ is adjacent to at least one vertex of v_k, $v_k \in \{D \cup C_1\}$, such that $\deg(v_k) \leq \deg(v_l)$ and v_l is adjacent to v_k. Then $D \cup C_1$ forms a minimal $\gamma_{wtb} - set$ of G. Clearly $|D \cup C_1| + |\deg(e)| \geq |\{H \cup H_1\}| + 2$, gives $\gamma_{wtb}(G) + \delta'(G) \geq \gamma_{wl}(G) + 2$.

Theorem 7: For any connected graph (p, q) graph G, with $p \geq 2$ vertices, then $\gamma_{wtb}(G) \leq \gamma_{sns}(G) + 1$.

Proof: Let $R = \{v_1, \dots, v_n\} \subseteq V(G)$ be the set of all endvertices in G and $S = \{v_1, \dots, v_m\} \subseteq V(G)$ be the set of all nonendvertices which are adjacent to the endvertices of G. Further, let $S_1 \subseteq V(G)$ be the set of nonendvertices which are not adjacent to endvertices of R and $S_1' \subseteq S_1$ be the set of vertices of G, such that $N[R \cup S_1'] = V(G)$. Hence $\{R \cup S_1'\}$ forms a minimal $\gamma - set$ of G. Further, if the induced subgraph $< V(G) - \{R \cup S_1'\} >$ is complete, then $\{R \cup S_1'\}$ itself is a $\gamma_{sns} - set$ of G. Otherwise, if $R = \phi$, then $\{S_1'\}$ is a $\gamma - set$ of G and if $\langle V(G) - S_1' \rangle$ is complete subgrraph, then $\{S_1'\}$ is also a $\gamma_{sns} - set$ of G. If not then select the set of vertices $\{v_j\}$ from $V - S_1'$, such that $< V(G) - S_1' \cup \{v_j\} >$ is a complete subgraph graph. Hence $[S_1' \cup \{v_j\}]$ is $\gamma_{sns} - set$ of G. Let $B = \{v_1, \dots, v_p\}$ be the set of block vertices in $T_b(G)$, such that $N[B] = V[T_b(G)]$. Hence B is a $\gamma - set$ of $T_b(G)$. Then for every $v_k \in V[T_b(G)] - B$ there is a vertex $v_l \in B$, such that $\deg(v_l) \leq \deg(v_k)$ with v_l is adjacent to v_k. Then B forms a minimal weak semitotal block dominating set of G. Therefore $|B| \leq |S_1' \cup \{v_j\}| + 1$, gives $\gamma_{wtb}(G) \leq \gamma_{sns}(G) + 1$.

Theorem 8: For any connected (p, q) graph G, with $p \geq 2$ vertices, then $\gamma_{wtb}(G) \leq \gamma_{ns}(G) + \alpha_0(G) - 1$.

Proof: Let $B = \{v_1, \dots, v_p\}$ be the vertex set of G. Suppose $D = \{v_1, \dots, v_q\} \subseteq B$ be a minimal dominating set of G. Further, if the induced subgraph $< V - D >$ is connected then D itself is a $\gamma_{ns} - set$ of G. Otherwise, there exists a vertex $w \in V(G) - D$ with $\deg(w) = 0$, then $D \cup \{w\}$ forms a nonsplit dominating set of G.

Let $C \subseteq B$ be the minimal set of vertices which covers all the edges of G with $|C| = \alpha_0(G)$. Let $S = \{v_1, \dots, v_n\} \subseteq V[T_b(G)]$ be the set of vertices with minimum degree and $\forall\, v_i \in S, 1 \leq i \leq n$ is adjacent to at least one vertex of $v_j \in V[T_b(G)] - S$, such that $N[S] = V[T_b(G)]$. Furthermore, if $\deg(v_i) \leq \deg(v_j)$ with $N(v_j) \cap S = \{v_i\}$. Hence S forms minimal $\gamma_{wtb} - set$ of G. Therefore $|S| \leq |D \cup \{w\}| + |C| - 1$, gives $\gamma_{wtb}(G) \leq \gamma_{ns}(G) + \alpha_0(G) - 1$.

The following theorem gives the lower bound for $\gamma_{wtb}(T)$ in terms of weak block domination number, Chromatic number and edge covering number of T.

Theorem 9: For any nontrivial tree T, $\gamma_{wtb}(T) \geq \gamma_{wb}(T) + \alpha_1(T) - \chi(T) - 1$.

Proof: Let $E = \{e_1, \dots, e_n\} \subseteq E(T)$ be the set of all endedges in T and $E_1 = E(T) - E$ be the set of all nonendedges which are not adjacent to the endedges of E, further, consider the subset $E_1' \subseteq E_1$. Then $\{E \cup E_1'\}$ be the minimal set of edges which covers all the vertices of T, such that $|E \cup E_1'| = \alpha_1(T)$.

Since the chromatic number of a tree $T, \chi(T) = 2$, then we now show that $\gamma_{wb}(T) + \alpha_1(T) - \chi(T) - 1 \leq \gamma_{wtb}(T)$. For any tree T, let $A = \{e_1, \dots, e_m\}$ be the set of edges in T. Let $K = \{v_1, \dots, v_k\} = V[B(T)]$ be the set of vertices corresponding to the edges of A. Let $J \subseteq K$ be the set of vertices, such that $N[J] = V[B(T)]$ and if, $\forall\, v_i \in J$ has degree at least two and $\forall\, v_j \in V[B(T)] - J$ with $deg(v_i) \leq deg(v_j)$ and $N(v_j) \cap J = \{v_i\}$. Then J forms a $\gamma_{wb} - set$ of T. Let $H = \{v_1, \dots, v_n\}$ be the set of all nonend block vertices in $T_b(T)$ with $\deg(v_i) = 2$, $\forall\, v_i \in H$ and let $H_1 = \{v_1, \dots, v_m\}$ be the set of all end block vertices in $T_b(T)$, such that $N[H \cup H_1] = V[T_b(T)]$. Then $D = \{H \cup H_1\}$ forms a minimal $\gamma -$ set of T. Further, let $H_2 = \{v_1, \dots, v_p\}$ be the set of all endvertices of T adjacent to end block vertices of H_1 in $T_b(T)$, then $\forall\, v_j \in V[T_b(T)] - \{H \cup H_2\}$ is adjacent to at least one vertex of $v_i \in \{H \cup H_2\}$ with $\deg(v_i) \leq \deg(v_j)$ and v_j is adjacent to v_i. Then $\{H \cup H_2\}$ forms a minimal weak semitotal block dominating set of T. Therefore $|H \cup H_2| \leq |J| + |E \cup E_1^{'}| - \chi(T) - 1$, gives $\gamma_{wtb}(T) \geq \gamma_{wb}(T) + \alpha_1(T) - \chi(T) - 1$.

Theorem10: For any nontrivial tree T, $\gamma_{wtb}(T) + \gamma^{'}(T) \geq \gamma_w(T) + \gamma_R(T) - 4$.

Proof: Let $D = \{v_1, \ldots, v_n\} \subseteq V(T)$ be the set of endvertices in T. Let $D_1 = V(T) - D$ be the set of minimum degree vertices in T and $D_1^{'} \subseteq D_1$, such that $N[D \cup D_1^{'}] = V(T)$. Furthermore, $\forall\, v_p \in V[T] - \{D \cup D_1^{'}\}$ is adjacent to at least one vertex $v_q \in \{D \cup D_1^{'}\}$ such that $\deg(v_q) \leq deg(v_p)$ and $N(v_p) \cap \{D \cup D_1^{'}\} = \{v_q\}$. Hence $D \cup D_1^{'}$ forms a minimal $\gamma_w - set$ of T. Let $E = \{e_1, \ldots, e_n\} \subseteq E(T)$ be the set of edges and $H \subseteq E$ be the minimal set of edges which covers all the edges of $E(T)$. Then H forms a $\gamma^{'} - set$ of T. Further, let a function $f : V(T) \to \{0, 1, 2\}$ and partition the vertex set $V(T)$ into $\{V_0$, $V1, V2$ induced by f with $Vi=ni$, for $i=0,\ 1,\ 2$. Suppose the set $V2$ dominates $V0$. Then $R=V1\cup V2$ forms a minimal Roman dominating set of T. For any tree T, each edge is a block. Let $A = \{B_1, \ldots, B_n\}$ be the blocks of T. Let $S = \{v_1, \ldots, v_m\}$ be the block vertices in $T_b(T)$ corresponding to the blocks of A, such that $|S| = \gamma_{wtb} - set$ of T. Further, suppose $S_1 \subseteq S$ be the end bock vertices adjacent to D, in $T_b(T)$. Then $\forall\, v_j \in V[T_b(T)] - \{S - S_1\} \cup D$ is adjacent to at least one vertex of $v_i \in \{S - S_1\} \cup D$, such that $\deg(v_i) \leq \deg(v_j)$ and v_j is adjacent to v_i. Thus $\{S - S_1\} \cup D$ forms a $\gamma_{wbt} - set$ of T. Hence $|\{S - S1\cup D| + |H| \geq |D\cup D1'| + |R| - 4$, gives $\gamma wtbT + \gamma' T \geq \gamma wT + \gamma RT - 4$.

Theorem 11: For any nontrivial tree T, then $\gamma_{wtb}(T) + 3 \geq \gamma_\varepsilon(T) + C$, where C is the cutvertices of T.

Proof: Let $C = \{c_1, \ldots, c_p\} \subseteq V(T)$ be the set of all cutvertices of T. Then $C_1 \subseteq C$ forms a $\gamma - set$ of T. Let $F \subseteq E(T)$ be the edge dominating set of T. Further, we consider $C_1^{'} \subseteq C_1$ and $F^{'} \subset F$, such that each element of T is adjacent or incident to the at least one element of $\{C_1^{'} \cup F^{'}\}$. Then $\{C_1^{'} \cup F^{'}\}$ forms an entire dominating set of T. In $T_b(T)$, let $B = \{v_1, \ldots, v_m\}$ be the set of block vertices which are at a distance 2, such that $N[B] = V[T_b(T)]$. Then $|B| = \gamma_{tb}(T)$. Suppose, let $A = \{v_1, \ldots, v_n\}$ be the set of end block vertices in $T_b(T)$ and $A_1 = \{v_1, \ldots, v_p\}$ be the set of endvertices adjacent to A, with $\deg(v_i) = 2$, $\forall\, v_i \in A_1$. Then $\forall\, v_j \in V[T_b(T)] - \{B - A\} \cup A_1$ is adjacent to at least one vertex of v_k, $\forall\, v_k \in \{B - A\} \cup A_1$, such that $\deg(v_k) \leq deg(v_j)$ and v_j is adjacent to v_k. Then $\{B - A\} \cup A_1$ forms a $\gamma_{wtb} - set$ of T. Clearly $|\{B - A\} \cup A_1| + 3 \geq |\{C_1^{'} \cup F^{'}\}| + |C|$, gives $\gamma_{wtb}(T) + 3 \geq \gamma_\varepsilon(T) + C$.

We established the following lower bound for $\gamma_{wtb}(G)$.

Theorem 12: For any connected $(p,\ q)$ graph G, with $p \geq 2$ vertices, then $\gamma_{wtb}(G) + 1 \geq \gamma_c(G) + \gamma(G^2)\ and\ G \neq C_p, p \geq 5$.

Proof: Suppose $G = C_p, p \geq 5$. Then by Theorem 1, $\gamma_{wtb}(C_p) + 1 \ngeq \gamma_c(C_p) + \gamma({C_p}^2)$, $p \geq 5$. Let $S = \{v_1, \ldots, v_n\} \subseteq V(G^2)$, such that $N[S] = V(G^2)$.Then S forms a minimal $\gamma - set$ of G^2. Let $D \subseteq V(G)$ be a minimal dominating set of G. Further, if the induced subgraph $< D >$ has one component then D itself is a connected dominating set of G. Otherwise, there exists a vertex set $\{v_i\} \epsilon\ V(G) - D$, such that $D \cup \{v_i\}$ is connected. Hence $D \cup \{v_i\}$ forms a connected dominating set of G. Let $B = \{B_1, \ldots, B_n\}$ be the blocks in G. Let $F = \{v_1, \ldots, v_m\}$ be the set of block vertices in $T_b(G)$ corresponding to the blocks of B, such that $N[F] = V[T_b(G)]$. Then F be the minimal dominating set of $T_b(G)$ and if, $\forall\, v_p \in F$ with $\deg(v_p) \leq \deg(v_q)$, $\forall\, \epsilon\, v_q\ V[T_b(G)] - F$ and v_q is adjacent to v_p. Then F forms a $\gamma_{wtb} - set$ of G. Thus $|F| + 1 \geq |D \cup \{v_i\}| + |S|$. Hence $\gamma_{wtb}(G) + 1 \geq \gamma_c(G) + \gamma(G^2)$. Otherwise, let $F^{'} = V[T_b(G)] - F$ be the set of vertices with minimum degree and satisfies the definition of weak dominating set. Then $F^{'} = \gamma_{wtb} - set$, gives $|F^{'}| + 1 \geq |D \cup \{v_i\}| + |S|$ so that $\gamma_{wtb}(G) + 1 \geq \gamma_c(G) + \gamma(G^2)$.

Theorem 13: For any connected (p,q) graph G, with $p \geq 2$ vertices, then $\gamma_{wtb}(G) \leq \gamma_e(G) + \gamma_p(G) + 2$.

Proof: Suppose A be the vertex set of G and let $A_1 \subseteq A$ be the set of all endvertices in G. Further, if $V_1 = V(G) - A_1$ and $A_1^{'} \subseteq V_1$, such that $N[A_1 \cup A_1^{'}] = V(G)$. Then $\{A_1 \cup A_1^{'}\}$ forms a minimal $\gamma_e - set$ of G. Suppose $S \subseteq V(G)$ is a minimal dominating set of G. If each vertex $v \in V(G) - S$ is dominated by exactly one vertex of S. Then S forms a perfect dominating set of G. Let $D = \{v_1, \dots, v_n\} \subseteq V[T_b(G)]$ be the set of vertices with minimum degree. Then D forms $a\ \gamma_{tb} - set$ of G, such that $\forall\ v_j \in V[T_b(G)] - D,\ deg(v_i) \leq deg(v_j),\ \forall\ v_i \in D$ with $N(v_j) \cap D = \{v_i\}$. Hence D forms a minimal weak semitotal block dominating set of G. Hence $|D| \leq |\{A_1 \cup A_1^{'}\}| + |S| + 2$, gives $\gamma_{wtb}(G) \leq \gamma_e(G) + \gamma_p(G) + 2$.

Theorem14: For any connected $(p,\ q)$ graph G, with $p \geq 3$ vertices, then $\gamma_{wtb}(G) \geq \gamma_s(G) + \beta_0(G) - 3$ and $G \neq W_p, C_p$ with $p \geq 8$.

Proof: For the graph $G = W_p,\ C_p$ with $p \geq 8,\ \gamma_{wtb}(G) < \gamma_s(G) + \beta_0(G) - 3$ and hence the result doesnot holds. Let $B = \{v_1, \dots, v_p\} \subseteq V(G)$ be the set of all endvertices and $C = \{v_1, \dots, v_q\} \subseteq V(G) - B$, further, consider the subset $C_1 \subseteq C$, such that $N(B) \cap N(C_1) = u \in V(G) - \{B \cup C_1\}$, then $\{B \cup C_1\}$ is an independent set with $|B \cup C_1| = \beta_0$. Suppose $D \subseteq V(G)$ be the minimal dominating set of G. Assume the induced subgraph $\langle V(G) - D \rangle$ has more than one component. Then D is a minimal split dominating set of G. Suppose $R = \{B_1, \dots, B_n\}$ be the blocks of G. Let $S = \{v_1, \dots, v_n\}$ be the block vertices in $T_b(G)$ corresponding to the blocks of R. Further, let S_1 be the set of vertices with minimum degree and $S_1^{'} \subseteq S_1$, such that $N[S \cup S_1^{'}] = V[T_b(G)]$. Then $S \cup S_1^{'}$ forms a $\gamma - set$ of $T_b(G)$ and if, $\forall\ v_j \in V[T_b(G)] - S \cup S_1^{'}$ is adjacent to at least one vertex of $v_i \in S \cup S_1^{'}$, such that $deg(v_i) \leq deg(v_j)$ Hence $S \cup S_1^{'}$ forms a minimal weak semitotal block dominating set of G. Thus $|S \cup S_1^{'}| \geq |D| + |B \cup C_1| - 3$, gives $\gamma_{wtb}(G) \geq \gamma_s(G) + \beta_0(G) - 3$.

Theorem15: For any nontrivial tree T, then $\gamma_{wtb}(T) \geq \gamma_c^{'}(T) + \beta_1(T) - 1$ and $T \neq P_p,\ p \geq 8$.

Proof: Suppose $T = P_p$ with $p \geq 8$. Then $\gamma_{wtb}(P_p) < \gamma_c^{'}(P_p) + \beta_1(P_p) - 1$. Let $F = \{e_1, \dots, e_n\} \subseteq E(T)$ be the set of all endedges and $F_1 = \{e_1, \dots, e_m\}$ be the set of edges adjacent to the endedges of F, $F_2 = E(T) - \{F \cup F_1\}$. Suppose $F_2^{'} \subseteq F_2$. Then for every element $e_j \in E(T) - \{F_1 \cup F_2^{'}\}$ is adjacent to at least one element $e_i \in \{F_1 \cup F_2^{'}\}$. Hence $S = \{F_1 \cup F_2^{'}\}$ forms a $\gamma^{'} - set$ of T. Further, if the induced subgraph $\langle S \rangle$ has one component then S is a minimal $\gamma_c^{'} - set$ of T. Let $A \subseteq E(T)$ be the maximal independent set with $|A| = \beta_1(T)$. Suppose $H = \{v_1, \dots, v_n\}$ be the block vertices in $T_b(T)$ corresponding to the blocks, $B = \{B_1, \dots, B_n\}$ of T. Since in semitotal block graph $T_b(T)$, each block is K_3, then $N[H] = V[T_b(T)]$. Then H is a $\gamma_{tb} - set$ of T. If H satisfies all the conditions of weak dominating set of T, then H is a $\gamma_{wtb} - set$ of T. Hence $|H| \geq |S| + |A| - 1$, gives $\gamma_{wtb}(T) \geq \gamma_c^{'}(T) + \beta_1(T) - 1$. It is also possible to satisfies the inequality of the theorem by considering. Suppose $H_1 = \{v_1, \dots, v_m\} \subseteq H$ be the set of end block vertices and $C \subseteq V(T)$ be the endvertices of T. Then $\forall\ v_l \in V[T_b(T)] - \{H - H_1\} \cup C$ is adjacent to at least one vertex of $v_k \in \{H - H_1\} \cup C$ with $deg(v_k) \leq deg(v_l)$. Hence $\{H - H_1\} \cup C$ is $\gamma_{wtb} - set$ of T. Thus $|\{H - H_1\} \cup C| \geq |S| + |A| - 1$, which gives $\gamma_{wtb}(T) \geq \gamma_c^{'}(T) + \beta_1(T) - 1$.

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